

# The Geometry of Renormalization

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## Abstract

This paper generalizes the equisingular connection developed by Connes and Marcolli. I find a global connection on the renormalization bundle that defines  $\beta$ -functions for regularization schemes whose regularization parameter parameterizes an infinitesimal disk and which satisfy the same locality condition that defines the equisingular connection. The advantage of the global connection is that it relates the  $\beta$ -functions of different regularization schemes by gauge transformations. Furthermore, it relates  $\beta$ -functions of different Lagrangians of the same type theory by gauge transforms, giving a context for looking at a  $\beta$ -function for a scalar field theory over a curved background space under  $\zeta$ -function regularization.

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## 1 Introduction

The process of regularization and renormalization is well known to physicists studying Quantum Field Theories (QFTs). There are several textbooks with good expositions on the processes of regularization and renormalization [13] chapter 8, [18] chapter 9, [19] chapters 18-21, and [21]. Broadly speaking, regularization is the process of rewriting an undefined quantity in terms of certain parameters such that the quantity is well defined away from a predetermined limit of the parameters. The renormalization is the process of making sense of the regularized quantity at the limit.

The dynamics of a QFT is defined by a Lagrangian density. The undefined quantities in this context are the values given by the Green's functions associated to this Lagrangian, where the Green's functions act on a fixed number of test functions. There are many different types of regularization processes commonly used, such as dimensional regularization, Pauli-Villars regularization, momentum cut-off regularization,  $\zeta$ -function regularization, and point splitting regularization to name a few commonly used schemes. The choice of which regularization scheme to apply to a QFT depends on the symmetries that need to be preserved for the QFT under study, and the ease of calculation of the scheme, among other factors. While the process

of regularization is well understood, very little is known about relationships between these regularization schemes.

Renormalization in the context of QFTs extracts a well defined physical value at a predetermined limit of the regularization parameters introduced. For massive fields, all the regularization schemes mentioned above have only one regularization parameter. There is a further requirement that this physical value match the value observed in experiment. A common method of renormalization for perturbative QFTs is called BPHZ renormalization. It was developed, corrected and proved in the 1950's and 1960's Bogoliubov, Parasiuk, Hepp, and Zimmermann as an algorithm for iteratively subtracting off divergent sub-interactions in the perturbative expansion of a given Green's function [3, 12, 22]. This renormalization method is useful when the regularization scheme rewrites the Green's functions in terms of Laurent series in the regularization parameter. While a well known algorithm in physics, BPHZ renormalization remained without a mathematical context until a break through paper by Connes and Kreimer in 2000 [4] showed that BPHZ renormalization for dimensional regularization of a scalar field theory is exactly the process of Birkhoff decomposition of loops defined by the regularization parameter in a connected complex Lie group associated to a Hopf algebra of Feynman graphs. Further work by Ebrahimi-Fard, Guo and Kreimer [8] shows that BPHZ renormalization for any regularization scheme that can be expressed as a Rota-Baxter algebra is equivalent to a Birkhoff decomposition problem. This was the beginning of the process of placing the problem of regularization and renormalization in a geometric context, and in trying to find a relation between different regularization schemes.

The key object necessary for finding physical values from a regularized QFT is called a  $\beta$ -function. It is defined by a differential equation which is part of the renormalization group equations formed by imposing certain physical conditions on the regularized Lagrangian. For details, see [19] chapter 21. Analytically, the  $\beta$ -function for any regularization scheme has not been solved for graphs containing more than a few loops. In [5], Connes and Kreimer find a way expressing the contributions of each graph in the Hopf algebra to the  $\beta$ -function of the QFT in terms of the Birkhoff decomposed loops. While this expression does nothing to help in the calculation of the  $\beta$ -function, it further opens the door for understanding the process of regularization geometrically, and understanding the relation between regularization schemes.

In 2006, [7] Connes and Marcolli introduced a renormalization bundle that geometrically represented the process of regularization and renormalization of Feynman integrals in a QFT. In particular, they worked with a scalar field theory and investigated dimensional regularization under BPHZ renormalization. The renormalization bundle  $P^* \rightarrow B^*$  is a  $G$  principal bundle, where  $G$  is the affine group scheme associated to the Hopf algebra of Feynman diagrams associated to the QFT developed by Connes and Kreimer in 2001 [4]. The base space  $B^* \simeq \mathbb{C}^\times \times \Delta^*$ , where  $\Delta^*$  is the punctured infinitesimal disk in  $\mathbb{C}$ . The sections of this bundle,  $(\gamma, t)$ , correspond to evaluators of regularized Lagrangians on Feynman diagrams associated to a QFT at a fixed energy level. Connes and Marcolli identify a class of connections on  $B^*$  associated to dimensional regularization which is defined uniquely by the  $\beta$ -function of the dimensionally regularized Lagrangian up to a gauge equivalence given by the group of holomorphic sections.

In this paper, I identify a global connection on  $P^* \rightarrow B^*$ . Key to defining this connection is a definition by Ebrahimi-Fard and Manchon [9] of a bijective correspondence,  $\tilde{R}$  between the Lie group  $G$  and its Lie algebra  $\mathfrak{g}$  defined by the grading,  $Y$ , on  $\mathcal{H}$ . In [14], Bondia, Ebrahimi-Fard and Patras show that  $\tilde{R}$  is closely related to the Dynkin operator on the Hopf algebra of Feynman graphs.

**Theorem 1.1.** *Let  $\omega$  be a connection on the bundle  $P^* \rightarrow B^*$  defined on sections of the bundle by the differential equation  $\gamma_t^* \omega = D\gamma_t(z)$ . The connection is uniquely defined on pullbacks defined by the sections by  $\tilde{R}(\gamma)$ .*

The advantage of defining a global connection is that it gives a way of understanding the relation between these sections. The choice of regularization scheme and Lagrangian corresponds to a gauge field, or pullback of this global connection over the section representing the choice. This has two important conceptual implication. Specifically, since sections of the renormalization bundle can be interpreted at evaluators of various regularized Lagrangians, a global connection gives a relationship between different regularization

schemes of the same Lagrangian represented as sections of this bundle, or between different Lagrangians, under the same regularization scheme that can be represented as sections of this bundle.

The first implications is that the global connection gives a way of conceptualizing a relationship between different regularization schemes. In fact, the equivalence class of connections defined by Connes and Marcolli in [7] defines when two regularization schemes should have the same  $\beta$ -function. On the other hand, a global connection relates two different  $\beta$ -functions by a gauge transformation. Furthermore, the global connection gives a way to relate between regularization schemes that do not yield a  $\beta$ -function and those that do, i.e. between non-renormalizable regularization schemes and renormalizable regularization schemes.

The second implications of the global connection is that there is now a way of understanding renormalization over curved space time. Currently, there is no global means of defining and renormalizing a QFT over a curved background space time manifold. Calculations are done in coordinate patches, and it is difficult to check for consistency of results across these patches. In the context of this renormalization bundle, two different coordinate patches will lead to two Lagrangians defining a QFT which have the same structure, but whose parameters may differ. The choice of regularization scheme and Lagrangian define two different sections of the renormalization bundle. Given a global connection on this bundle, this choice of section is equivalent to a choice of gauge. The existence of a global connection over the renormalization bundle provides a way of comparing the results after renormalization. It also shows that BPHZ renormalization is consistent over a general manifold.

Section two of this paper reviews the development of the tools necessary for the construction of the renormalization bundle, following [4], [6], [9]. Section 3 discusses the physical and geometrical  $\beta$ -function, following [21] and [6]. Section 4 defines the global section on the renormalization bundle.

## 2 The Connes Marcolli renormalization bundle

### 2.1 Feynman diagrams

In this paper, I work with the renormalizable scalar quantum field theory of valence 3 interactions. It can be defined by the Lagrangians of the form

$$\mathcal{L} = \frac{1}{2}(|d\phi|^2 - m^2\phi^2) + g\phi^3, \quad (1)$$

where  $m$  is the mass of the parameter, and  $g$  is the coupling constant. I use this particular Lagrangian to stay consistent with the work in [4], [5] and [7]. Later in this paper, I discuss what happens if I consider a class of Lagrangians obtained by allowing  $\phi$ ,  $m$  and  $g$  to vary. One can derive the equations of motion for interactions involving scalar fields mass  $m$ . These interactions can be depicted graphically in Feynman diagrams.

**Definition 1.** A Feynman diagram is an abstract representation of an interaction of several fields. It is drawn as a connected, not necessarily planar, graph with possibly differently labeled edges. It is a representative element of the equivalence class of planar embeddings of connected non-planar graphs. The types of edges, vertices, and the permitted valences are determined by the Lagrangian of the theory in the following way:

1. The edges of a diagram are labeled by the different fields in the Lagrangian. For this Lagrangian, there is only one type of edge.
2. The composition of monomial summands with degree  $> 2$  in the Lagrangian density correspond to permissible valences and composition of vertices of the Feynman diagrams. The  $\phi^3$  term means that all vertices have valence 3.
3. Vertices of valence one are replaced by half edges and called external edges. All other edges are internal.

The building blocks of these Feynman diagrams are called one particle irreducible (1PI) diagrams.

**Definition 2.** A 1PI graph is a connected Feynman graph such that the removal of any internal edge still results in a connected graph.

All Feynman diagrams associated to a theory can be constructed by gluing together 1PI diagrams along an exterior edge.

The Feynman rules are an algorithm that map Feynman diagrams to integral operators that act on a space of external momentum data, called Feynman integrals. Details about the Feynman rules and Feynman integrals can be found in textbooks such as [17], [19] and [20]. These integral operators associated to Feynman diagrams are frequently undefined. In order to make sense of the interactions, the operators need to be *regularized*, or written in terms of an extra parameter such that the integrals are defined away from a fixed limit. Regularization gives a one parameter family of well defined operators, but they do not represent the interactions for the original Lagrangian. To interpret the results of the original theory, *renormalize* the regularized theory to extract finite values at the limit.

Connes and Marcolli [7] construct their renormalization bundle in terms of dimensional regularization, but I require only that the regularization scheme lets one rewrite the Feynman integrals as operators from the external momentum data to Laurent polynomials with finite poles.

The renormalization scheme used in the Connes Marcolli renormalization bundle is BPHZ renormalization, which is an algorithm developed by Bogoliubov, Hepp, Parasiuk and Zimmermann in the 1950s and 60s that iteratively subtracts off subdivergences associated to subgraphs [3, 12, 22].

For the BPHZ renormalization procedure to work on a QFT, it has to be *renormalizable*. This condition means that the degree of divergence of the theory are controlled, in that the degree of divergent of divergence of a Feynman diagram does not increase with the complexity of a graph. This is equivalent to the statement that the space-time dimension of the QFT associated to the Lagrangian in (1) is 6, and that the 1PI Feynman diagrams only have 2 or 3 external legs. For details about degrees of divergence of Feynman diagrams, see [13] section 8.1 or [21].

BPHZ renormalization identifies subdivergences in a Feynman diagram,  $\Gamma$ , associated to divergent Feynman diagrams embedded inside it, and iteratively subtracts off the subdivergences. The graphs associated to these subdivergences,  $\gamma$  are called subgraphs of  $\Gamma$ .

**Definition 3.** Let  $V(\Gamma)$  be the set of vertices of a graph  $\Gamma$ ,  $I(\Gamma)$  the set of internal edges and  $E(\Gamma)$ , the set of external edges. The Feynman diagram  $\gamma$  is an admissible subgraph of a 1PI Feynman diagram  $\Gamma$  if and only if the following conditions hold:

1. The Feynman diagram  $\gamma$  is a divergent 1PI Feynman diagram, or a disjoint union of such diagrams. If  $\gamma$  is connected, it is a connected admissible subgraph, otherwise it is a disconnected admissible subgraph. I use admissible subgraph to mean both unless otherwise specified.
2. Let  $\gamma' = \gamma \setminus E(\gamma)$  be the graph of  $\gamma$  without its external edges. There is an embedding  $i : \gamma' \hookrightarrow \Gamma$  such that the field labels of each internal edge and the valence of each vertex of  $\gamma$  are preserved.
3. Let  $f_v$  be the set of legs (internal and external) meeting the vertex  $v \in V(\gamma)$ . Then  $f_{i(v)} \subset I(\Gamma) \cup E(\Gamma)$  has the same number of each type of leg as  $f_v \subset I(\gamma) \cup E(\gamma)$ .

The last condition ensures that the external leg conditions are preserved under the embedding. Finally, I need a definition of a contracted graph to represent the divergences that remain after the subtraction of the subdivergences.

**Definition 4.** Let  $\gamma$  be a disconnected admissible subgraph of  $\Gamma$  consisting of the connected components  $\gamma_1 \dots \gamma_n$ . A contracted graph  $\Gamma // \gamma$ , is the Feynman graph derived by replacing each connected component  $i(\gamma'_j)$ , with a vertex  $v_{\gamma_j} \in V(\Gamma // \gamma)$ .

This subgraph and contracted graph structure gives rise to a Hopf algebra structure on the Feynman diagrams.

## 2.2 Hopf algebra

A Hopf algebra can be built out of the Feynman diagrams by assigning variables  $x_\Gamma$  to each 1PI graph  $\Gamma$  and considering the polynomial algebra on these variables  $\mathcal{H} = \mathbb{C}[\{x_\Gamma | \Gamma \text{ is 1PI}\}]$ . This Hopf algebra is constructed in [4]. The product of two variables in this algebra  $m(x_{\Gamma_1} \otimes x_{\Gamma_2}) = x_{\Gamma_1 \Gamma_2}$  corresponds to the disjoint union of graphs, and the unit is given by the empty graph,  $1_{\mathcal{H}} = x_\emptyset$ . The coproduct of this Hopf algebra is given by the subgraph and contracted graph structure of the Feynman diagrams

$$\Delta x_\Gamma = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\gamma \subsetneq \Gamma} x_\gamma \otimes x_{\Gamma//\gamma}$$

where the sum is taken over all proper admissible subgraphs of  $\Gamma$ . The kernel of the co-unit is the ideal generated by all  $x_\Gamma$  such that  $\Gamma$  is non-empty. The antipode is defined to satisfy the antipode condition for Hopf algebras

$$\begin{aligned} S : \mathcal{H} &\rightarrow \mathcal{H} \\ x_\Gamma &\rightarrow -x_\Gamma - \sum_{\gamma \subset \Gamma} m(S(x_\gamma) \otimes x_{\Gamma//\gamma}) . \end{aligned}$$

This is a bigraded Hopf algebra, with one grading given by loop number and the other by insertion number. Details on the two grading structures are given in [4] and [2]. This Hopf algebra is associative, co-associative and commutative, but not co-commutative.

In general, Hopf algebras can be interpreted as a ring of functions on a group. Since the spectrum of a commutative ring is an affine space, the group in question is affine group scheme,  $G = \text{Spec } \mathcal{H}$ . The group laws on the Lie group  $G$  are covariantly defined by the Hopf algebra properties

$$\begin{aligned} (\text{id} \otimes \Delta)\Delta &= (\Delta \otimes \text{id})\Delta \quad \leftrightarrow \quad \text{multiplication} \\ (\text{id} \otimes \varepsilon)\Delta &= \text{id} \quad \leftrightarrow \quad \text{identity} \\ m(S \otimes \text{id})\Delta &= \varepsilon\eta \quad \leftrightarrow \quad \text{inverse} \end{aligned}$$

The group  $G$  can also be viewed as a functor from a  $\mathbb{C}$  algebra  $A$  to  $G(A) = \text{Hom}_{\text{alg}}(\mathcal{H}, A)$ . The affine group scheme  $G$  is developed in detail in [1]. The last condition above means that if  $\gamma \in G(A)$ , and  $x \in \mathcal{H}$ , then  $\gamma^{-1}(x) = S(\gamma(x)) = \gamma(S(x))$ .

The Lie algebra  $\mathfrak{g}$  associated to  $G$  is generated by the algebra homomorphisms given by the Kroniker  $\delta$  functions on the generators of  $\mathcal{H}$ . That is,  $\delta_\gamma$  is a generator of  $\mathfrak{g}$  if and only if

$$\delta_{\gamma_1}(x_{\gamma_2}) = \begin{cases} 1 & \gamma_1 = \gamma_2, \\ 0 & \text{else.} \end{cases}$$

By the Milnor-Moore theorem, the universal enveloping algebra is isomorphic to the restricted dual of  $\mathcal{H}$

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{H}^\vee = \oplus_l \mathcal{H}^{l*}$$

where the grading is given by the loop number of the graph. The restricted dual is the direct sum of the duals of each graded component of  $\mathcal{H}$ . The product is defined on  $\mathcal{H}^\vee$  by the convolution product

$$\alpha_1 \star \alpha_2(x_\Gamma) = m(\alpha_1 \otimes \alpha_2)(\Delta x_\Gamma) \quad \alpha_i \in \mathcal{H}^\vee .$$

This is described in detail in [4] and [16]. The convolution product on  $\mathfrak{g}$  acts as an insertion operator on  $\mathcal{H}$ . For two generators of  $\mathcal{H}$ ,  $x_{\Gamma_1}$  and  $x_{\Gamma_2}$ , define

$$x_{\Gamma_1} \star x_{\Gamma_2} = \sum_{x_\Gamma} m(\delta_{\Gamma_1} \otimes \delta_{\Gamma_2})(\Delta x_\Gamma) \cdot x_\Gamma$$

where the sum is taken over all generators of  $\mathcal{H}$ . This product induces an insertion product on the 1PI graphs of a theory in the same fashion that the coproduct on  $\mathcal{H}$  is induced by the subgraph structure on the 1PI graphs. This convolution product induces a pre Lie structure on the generators on the 1PI graphs of a theory. The Lie bracket

$$[x_{\Gamma_1}, x_{\Gamma_2}] = x_{\Gamma_1} \star x_{\Gamma_2} - x_{\Gamma_2} \star x_{\Gamma_1}$$

follows the Jacobi identity, as can be checked. For details on this construction, see [4] and [10].

Manchon [15] develops bijective correspondence between  $G(A)$  and a  $\mathfrak{g}(A)$  defined as

$$\begin{aligned} \tilde{R} : G(A) &\rightarrow \mathfrak{g}(A) \\ \gamma &\mapsto \gamma^{\star-1} \star Y(\gamma) . \end{aligned}$$

Manchon also shows that this is inverse of the time ordered expansional defined by Connes and Marcolli in [1]

$$\begin{aligned} Te : \mathfrak{g}(A) &\rightarrow G(A) \\ \alpha &\mapsto Te^{\int_a^b \theta_{-s}(\alpha) ds} \end{aligned}$$

*Remark 1.* The time ordered expansional is not the same bijection as that of the standard exponential map from  $\mathfrak{g}$  to  $G$ ,

$$\exp(\alpha) = \epsilon + \alpha + \frac{\alpha^2}{2!} + \dots$$

The time ordered expansional is given by the formula

$$Te^{\int_a^b \theta_{-s}(\alpha) ds} = \epsilon + \sum_{n=1}^{\infty} \underbrace{Y^{-1}(\dots Y^{-1}(\alpha) \dots)}_{n \text{ times}} .$$

In fact the operator  $S \star Y(\gamma)$  is closely related to the Dynkin operator on the commutative Hopf algebra  $\mathcal{H}$  of Feynman graphs [14].

## 2.3 Birkhoff decomposition and Rota-Baxter algebras

In [4], Connes and Kreimer show that BPHZ renormalization can be written as a composition of loops in the Lie group  $G$  using the Birkhoff decomposition theorem. Connes and Marcolli also explain this construction in [1].

Let  $\mathcal{A} = \mathbb{C}\{\{z\}\}$  the the algebra of formal Laurent series in  $z$  with poles of finite order. Then  $\text{Spec } \mathcal{A} = \Delta^*$ , the punctured infinitesimal disk around the origin in  $\mathbb{C}$ . If  $\gamma(z)$  is a map from a simple loop not containing the origin in  $\Delta^*$  to  $G$ , by the Birkhoff decomposition theorem,  $\gamma(z)$  decomposes as the product

$$\gamma(z) = \gamma_-^{-1}(z) \gamma_+(z) ,$$

where  $\gamma_+(z)$  is a well defined map in the interior of the loop (containing  $z = 0$ ), and  $\gamma_-^{-1}(z)$  is a well defined map outside of the loop (away from  $z = 0$ ). This decomposition is uniquely defined by choosing a normalization on  $\gamma_-^{-1}$ .

There is a natural isomorphism between elements of the group  $G(\mathcal{A})$  and the maps

$$\gamma(z) : \Delta^* \rightarrow G .$$

Therefore each  $\gamma(z)$  can be written as a Laurent series with poles of finite order and coefficients in  $G(\mathbb{C})$  convergent in  $\Delta^*$ . The algebra homomorphisms decompose as  $\gamma(z) = \gamma_-^{\star-1}(z) \star \gamma_+(z)$ , where  $\star$  is the product on  $\mathcal{H}^\vee$  and  $G(A)$ . Following [4], the normalization for the uniqueness of the Birkhoff decomposition is given by  $\gamma_-^{\star-1}(z)(1) = 1$ .

*Remark 2.* Introducing a complex parameter  $z$ , is equivalent to introducing a regulator to a QFT. Each  $\gamma(z)(x_\Gamma)$  corresponds to the evaluation of the graph  $\Gamma$  under a regularized Lagrangian under a regularization scheme that yields results in  $\mathbb{C}\{\{z\}\}$ .

Since  $\gamma_+(z)(x_\Gamma)$  is well defined at  $z = 0$ , it can be written as a somewhere convergent formal power series in  $z$ . Rewrite  $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ , where  $\mathcal{A}_+ = \mathbb{C}\{z\}$ . This is equivalent to saying that  $\gamma_+(z) \in G(\mathcal{A}_+)$ . That is, for  $x_\Gamma \in \mathcal{H}$ ,  $\gamma_+(z)(x_\Gamma)$  is a holomorphic function in  $z$ . By the normalization condition on the decomposition, for  $x_\Gamma \in \mathcal{H}^0$ , that is, if it is a constant multiple of  $x_\emptyset$ ,  $\gamma_-^{\star-1}(z) \circ \epsilon(x_\Gamma) = \epsilon(x_\Gamma)$ . If  $x_\Gamma \in \ker \epsilon$ , then  $\gamma_-^{\star-1}(z)(x_\Gamma) \in \mathcal{A}_-$ . That is, for  $x_\Gamma \in \ker \epsilon$ ,  $\gamma_-^{\star-1}(z)(x_\Gamma)$  is a Laurent series containing only negative powers of  $z$ .

Connes and Kreimer [4] show that the recursive formula for calculating  $\gamma_+(z)(x_\Gamma)$  and  $\gamma_-(z)(x_\Gamma)$  is the exact same as the recursive formula for calculating the renormalized and counterterm contributions respectively of a Feynman diagram  $\Gamma$  to the regularized Lagrangian given by BPHZ. Associating  $\gamma(z)$  to a regularization scheme associates  $\gamma_+(z)$  to the process of extracting renormalized values and  $\gamma_-(z)$  to determination of the counterterm of  $\Gamma$ . This method works for any regularization scheme that results in regularized Feynman integrals as operators from the external momentum data to  $\mathbb{C}\{\{z\}\}$ .

Work by Ebrahimi-Fard, Guo and Kreimer [8] extends the class of regulation schemes that can be decomposed into loops corresponding to regularized and counterterm evaluations. They show that BPHZ type renormalization can be applied to any regularization scheme that results in regularized Feynman integrals as operators from the external momentum data to Rota-Baxter algebras.

**Definition 5.** A Rota-Baxter algebra is a pair  $(A, P)$  consisting of an  $k$ -algebra  $A$  and a  $k$ -linear operator  $P$  on  $A$  such that

$$P(x)P(y) + \theta P(xy) = P(xP(y)) + P(P(y)x)$$

where  $x, y \in A$  and  $\theta \in \mathbb{R}$  is weight of the operator. If  $P$  is a Rota-Baxter operator, then so is  $\mathbb{I} - P$ , for  $\mathbb{I}$  the identity operator.

**Example 1.** Integration is a Rota-Baxter operator of weight 0 on the algebra  $A = \text{Cont}(\mathbb{R}, \mathbb{R})$ , continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Define the integration operator as  $I(f(x))$

$$I(f(x)) = \int_0^x f(t)dt .$$

To see that this is a Rota Baxter operator, notice that integration by parts can be rewritten

$$I(f(x)I(g(x))) = I(f(x))I(g(x)) - I(I(f(x))g(x)) .$$

Ebrahimi-Fard, Guo and Kreimer show that if  $A$  is a complete filtered algebra, and  $\gamma \in \text{Hom}(\mathcal{H}, A)$ , then the Birkhoff decomposition is defined by the Rota-Baxter operator  $P$  as above, replacing  $\pi$  with  $P$ .

### 3 The $\beta$ -function

The regularization process results in a Lagrangian that is a function of the regularization parameter. Prior to regularization, the Lagrangian of any theory is scale invariant. That is

$$\int_{\mathbb{R}^n} \mathcal{L}(x) d^n x = \int_{\mathbb{R}^n} \mathcal{L}(x) d^n(tx) .$$

When the Lagrangian is regularized, and written in terms of a regularization parameter,  $z$ , it is no longer scale invariant. Specifically, the counterterms of a theory depends on the scale of the Lagrangian. In order to preserve scale invariance in the regularized Lagrangian one introduces a regularization mass, which is also a function of the regularization parameter, to cancel out any scaling effects introduced by regularization.

### 3.1 Derivation in physics

The renormalization group describes how the dynamics of Lagrangian depends on the scale at which it is probed. One expects that probing at higher energy levels reveals more details about a system than at lower energies. To go from higher energy to lower, average over the extra information at the higher energy,  $\lambda$ , and rewrite it in terms of a finite number of parameters at a lower energy,  $\mu$ . The Lagrangian at the lower energy scale is called the *effective Lagrangian at  $\mu$* ,  $(\mathcal{L}, \mu)$ . For a specified set of fields and interactions the effective Lagrangian at a  $\mu$  is a Lagrangian with coefficients which depend on the scale,  $\mu$ .

Formally, let  $M \simeq \mathbb{R}_+$  be a non-canonical energy space, with no preferred element. Fix a set of fields and interactions. Call  $S$  the set of effective Lagrangians for this system in the energy space,  $M$ . For  $\lambda, \mu \in M$  such that  $\lambda > \mu$ , there is a map

$$R_{\lambda, \mu} : S \rightarrow S \quad (2)$$

so that the effective Lagrangian at  $\mu$  is written  $R_{\lambda, \mu} \mathcal{L}$  for  $\mathcal{L} \in S$ . The map in (2) can be written as an action of  $(0, 1]$  on  $S \times M$ :

$$\begin{aligned} (0, 1] \times (S \times M) &\rightarrow S \times M \\ t \circ (\mathcal{L}, \lambda) &\mapsto (R_{\lambda, t\lambda} \mathcal{L}, t\lambda) . \end{aligned} \quad (3)$$

In the effective Lagrangian  $R_{\lambda, t\lambda} \mathcal{L}(t)$ , all parameters,  $m$ ,  $\phi$ , and  $g$  are functions of the mass scale  $t$ . The map  $R_{\lambda, \mu}$  satisfies the properties

1.  $R_{\lambda, \mu} R_{\mu, \rho} = R_{\lambda, \rho}$ .
2.  $R_{\lambda, \lambda} = 1$ .

**Definition 6.** The set  $\{R_{\lambda, \mu}\}$  forms a semi-group called the renormalization group in the physics literature.

The renormalization group equations can be derived from differentiating the action in (3) and solving

$$\frac{\partial}{\partial t} (R_{\lambda, t\lambda} \mathcal{L}_{ct}) = 0 . \quad (4)$$

This differential equation gives rise to a system of differential equations that describe the  $t$  dependence of the unrenormalized parameters,  $m(t)$ ,  $g(t)$  and  $\phi(t)$ , in  $R_{\lambda, t\lambda} \mathcal{L}(t)$ . To solve the renormalization group equations, it is sufficient to solve for  $g(t)$ . The  $\beta$ -function describes the  $t$  dependence of  $g$  and can be written as

$$\beta(g(t)) = t \frac{\partial g(t)}{\partial t} .$$

This above development of the renormalization group and renormalization group equations follows [11]. For details on the renormalization group equations for a  $\phi^4$  theory, QED and Yang-Mills theory, see [19] chapter 21 or [18] Chapter 9.

**Example 2.** The  $\beta$ -functions listed below are calculated in terms of a power series in the coupling constant. The following are the one loop approximations of the  $\beta$ -functions for various theories. [19] [21]

1. For the scalar  $\phi^4$  theory in 4 space-time dimensions,

$$\beta(g) = \frac{3g^2}{16\pi^2} .$$

For a scalar  $\phi^3$  theory in 6 space-time dimensions,

$$\beta(g) = \frac{-g^3}{128\pi^3} .$$

2. For QED, the  $\beta$ -function has the form

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5)$$

where  $e$  is the electric charge.

3. For a general Yang-Mills theory with symmetry group  $G$ , the  $\beta$ -function has the form

$$\beta(g) = -\frac{11g^3}{48\pi^2}C_2(G)$$

where  $C_2$  is the quadratic Casimir operator.

4. For QCD, the  $\beta$ -function has the form

$$\beta(g) = -\frac{1}{48\pi^2}(33 - 2N_f)g^3$$

where  $N_f$  is the number of fermions.

Connes and Marcolli show that the  $\beta$ -function of a renormalization theory is an element of  $\mathfrak{g}$ . The quantities listed above are the sums of the  $\beta$  function evaluated on the one loop graphs. That is, the geometric  $\beta$ -function for a section  $\gamma$  is given by

$$\beta(\gamma) = \sum_{x_\Gamma} \beta(\gamma)(x_\Gamma) ,$$

where the sum is taken over the  $x_\Gamma$  generating  $\mathcal{H}^1$ .

### 3.2 As a geometric object

The geometric  $\beta$ -function requires a more general construction of the renormalization group and effective Lagrangians. In the renormalization bundle, the non-canonical energy space is given by  $M \simeq \mathbb{C}^\times$ . The space  $S$  of effective Lagrangians is replaced by the space  $G(\mathcal{A})$ , the space of evaluators of a regularized effective Lagrangians. The renormalization group is a group in this generalization (not just a semi group) given by  $\theta_s = e^{sY}$  for  $s \in \mathbb{C}$ . The action of the renormalization group can be written as a  $\mathbb{C}^\times$  action that factors through  $\mathbb{C}$  by setting  $t(s) = e^s$

$$\begin{aligned} t^Y : G(\mathcal{A}) &\rightarrow G(\mathcal{A}) \\ \gamma(z) &\mapsto t^Y \gamma(z) = \gamma_t(z) . \end{aligned}$$

The space  $S \times M$  becomes  $\tilde{G}(\mathcal{A}) = G(\mathcal{A}) \rtimes_{\theta} \mathbb{C}^\times$  in the notation of [1]. The action of  $\mathbb{C}^\times$  on  $\tilde{G}(\mathcal{A})$  is given by

$$\begin{aligned} \mathbb{C}^\times \times \tilde{G}(\mathcal{A}) &\rightarrow \tilde{G}(\mathcal{A}) \\ t \circ (\gamma, \lambda) &\mapsto (t^Y \gamma, t\lambda) . \end{aligned} \tag{5}$$

The renormalization bundle,  $P^* \rightarrow B^*$  is a  $\tilde{G}(\mathcal{A})$  principle bundle. By the  $\mathbb{C}^\times$  action in (5), it is a  $\mathbb{C}^\times$  invariant bundle. The base space  $B^* \simeq \Delta^* \times \mathbb{C}^\times$  is a product of the regularization parameter and the non-canonical energy space. In this context, the  $\beta$ -function is given by

$$\beta(\gamma(z)) = \frac{d}{dt} \Big|_{t=1} \lim_{z \rightarrow 0} \gamma(z)^{\star-1} \star t^{zY}(\gamma(z)) .$$

This is only well defined when  $\gamma(z)$  satisfies condition (4). To find the derivation of the geometric  $\beta$ -function in this context, see [5], [1] or [9].

## 4 Equisingular connections

This section develops a global connection on the Connes-Marcocoli renormalization bundle. The connections on  $B^*$  identified by Connes and Marcolli in [1] correspond to the pullbacks of a single global connection on  $P^*$  over sections  $\gamma_t$ .

Let  $\omega$  be a connection on  $P^*$ , defined on its pullbacks by the logarithmic differential operator, as in [1].

**Definition 7.** Let  $D$  be a differential operator.

$$\begin{aligned} D : \tilde{G}(\mathcal{A}) &\rightarrow \Omega^1(\tilde{\mathfrak{g}}) \\ (\gamma(z), t) &\mapsto (\gamma(z), t)^{\star-1} \star d(\gamma(z), t) . \end{aligned}$$

Connes and Marcolli use this operator to define a connection corresponding to a particular section  $\gamma_\mu(z)$  representing the function evaluating 1PI Feynman diagrams defined by a dimensionally regularized Lagrangian at the energy scale  $\mu$  [1]. Many of the properties of that section extend to the entire connection.

**Lemma 4.1.** For  $f, g \in \tilde{G}(\mathcal{A})$ , the differential  $D(f) = f^*\omega$  defines a connection on section  $f$  of  $P^* \rightarrow B^*$ .

*Proof.* If  $D$  defines a connection, it must satisfy equation

$$(f^{\star-1} \star g)^*\omega = g^{-1}dg + g^{\star-1}(f^*\omega)g , \quad (6)$$

for  $f, g \in \tilde{G}(\mathcal{A})$ . Since  $df^{-1} = -f^{-1}df f^{-1}$ ,

$$D(f^{-1}g) = Dg - g^{-1}f f^{-1}df f^{-1}g ,$$

or

$$Dg = D(f^{-1}g) + (f^{-1}g)^{-1}Df(f^{-1}g) .$$

which satisfies equation (6). □

**Proposition 4.2.** The connection  $\omega$  is  $\mathbb{C}^\times$  equivariant,

$$u^Y \omega(z, t, x) = \omega(z, ut, u^Y x) .$$

*Proof.* The proof given in [1] of this statement for  $\gamma_\mu^*\omega$  generalizes to all sections  $(\gamma(z), t)$ , and thus to the entire connection. Since  $P^* \rightarrow B^*$  is a  $\mathbb{C}^\times$  equivariant bundle. □

Since  $(\gamma(z), t) = t \circ (t^{-Y}\gamma(z), 1)$ , by the  $\mathbb{C}^\times$  action on  $\tilde{G}(\mathcal{A})$ , and  $(t^Y\gamma(z), 1)$  is identified with  $\gamma_t$ , it is sufficient to define the connection of sections of the form

$$(t^Y\gamma(z), 1)^*\omega = \gamma_t^*\omega .$$

**Proposition 4.3.** Given any section  $\gamma_t$ , one can directly calculate the corresponding pullback of the connection  $\omega$  on it.

$$D(t^Y\gamma(z)) = t^Y(\gamma^{\star-1}(z) \star \partial_z \gamma(z))dz + t^Y(\tilde{R}(\gamma)(z))\frac{dt}{t} .$$

*Proof.* One has

$$d(t^Y\gamma(z)) = t^Y(\partial_z \gamma(z))dz + t^Y\gamma(z) \star \tilde{R}(\gamma(z))\frac{dt}{t} .$$

Multiplying on the left by  $\gamma^{\star-1}$  gives the logarithmic derivative

$$Dt^Y\gamma(z) = t^Y(\gamma^{\star-1}(z) \star \partial_z \gamma(z))dz + t^Y(\tilde{R}(\gamma)(z))\frac{dt}{t} .$$

□

Since  $\omega \in \Omega^1(\tilde{\mathfrak{g}})$ ,  $\omega$  has the form

$$\begin{aligned}(\gamma, t)^*\omega &= a_\gamma(z, t)dx + b_\gamma(z, t)\frac{dt}{t} \\ (\gamma_t^*)\omega &= a_{\gamma_t}(z, 1)dx + b_{\gamma_t}(z, 1)\frac{dt}{t}.\end{aligned}$$

The terms  $a_{\gamma_t}$  and  $b_{\gamma_t}$  are defined as

$$\begin{aligned}a_{\gamma_t}(z, 1) &= t^Y(\gamma^{*-1}(z) \star \partial_z \gamma(z)) \\ b_{\gamma_t}(z, 1) &= t^Y(\tilde{R}(\gamma)(z)).\end{aligned}\tag{7}$$

**Proposition 4.4.** *The connection  $\omega$  is flat.*

*Proof.* It is sufficient to check that each pullback is flat. That is, that all the pullbacks satisfy

$$[a_{\gamma_t}(z, 1), b_{\gamma_t}(z, 1)] = \partial_t(a_{\gamma_t}(z, 1)) - \partial_z(b_{\gamma_t}(z, 1)).\tag{8}$$

□

Now I can state the main theorem of this paper.

**Theorem 4.5.** *Let  $\omega$  be a connection on the bundle  $P^* \rightarrow B^*$  defined on sections of the bundle by the differential equation  $\gamma_t^*\omega = D\gamma_t(z)$ . The connection is defined by  $\tilde{R}(\gamma) \in \mathfrak{g}(\mathcal{A})$ .*

*Proof.* Let  $\omega$  be a connection on  $P^* \rightarrow B^*$ . Since  $\omega$  is  $\mathbb{C}^\times$  invariant, it is sufficient to consider pullbacks of  $\omega$  along sections  $(\gamma_t, 1)$ , which can be written

$$\gamma_t^*\omega = a_{\gamma_t}(z, 1)dx + b_{\gamma_t}(z, 1)\frac{dt}{t}.$$

If

$$a_{\gamma_t}(z, 1) = Te^{-\int_0^\infty \theta_{-s}t^{-Y}b_{\gamma_t}(z, 1)ds} \star \partial_z Te^{\int_0^\infty \theta_{-s}t^{-Y}b_{\gamma_t}(z, 1)ds}$$

then

$$\gamma_t^*\omega = DTe^{\int_0^\infty \theta_{-s}t^{-Y}b_{\gamma_t}(z, 1)ds}.$$

Conversely, defining the connection by the pullback of its sections, we see that  $\gamma^*\omega$  is uniquely defined by  $\gamma$ . The section  $\gamma$  is uniquely defined by the map  $\tilde{R}(\gamma) \in \mathfrak{g}(\mathcal{A})$ . □

Theorem 4.5 is a generalization of the main result of Connes and Marcolli in [7]. They use the fact that for a flat connection,  $Te^{\int_\rho \omega(z, t)dt}$  is determined only by the homotopy class of the path,  $\rho$ , over which the integral is taken, to uniquely define the pullback  $\gamma^*\omega$  for a specific section.

Defining the connection on  $P^* \rightarrow B^*$  by the map  $\tilde{R}$  as in Theorem 4.5 loses the geometric intuition for the connection found in Connes and Marcolli's definition. However, unlike the Connes-Marcolli definition, this construction defines a connection globally on  $P^* \rightarrow B^*$ . Theorem 4.5 defines a connection for all regularization schemes of Lagrangians represented by sections of the bundle  $P^* \rightarrow B^*$ .

For instance, the evaluation of Feynman graphs of a Lagrangian under  $\zeta$ -function evaluation and dimensional regularization both yield results in  $\mathbb{C}\{\{z\}\}$ . Therefore, both regularization schemes can be represented as different sections of the  $P^* \rightarrow B^*$  bundle.

In fact, using the Rota-Baxter technology developed in [8], this renormalization bundle, and all subsequent calculations on can be carried out for any Lagrangian with a regularization scheme that evaluates Feynman diagrams into a complete, filtered Rota-Baxter algebra.

Connes and Marcolli [7] continue to define the  $\beta$ -function under dimensional regularization by defining an equivalence class on these pullbacks defined by the group of gauge transformations  $G(\mathcal{A}_+)$ . The sections corresponding to counterterms are left invariant under this equivalence relation.

**Definition 8.** Two pullbacks of the connection  $\gamma_t^* \omega$  and  $\gamma_t'^* \omega$  are equivalent if and only if one pullback can be written in terms of the action of  $G(\mathcal{A}_+)$  on the other

$$\gamma_t'^* \omega = D\psi_t + \psi_t^{*-1} \star \gamma_t^* \omega \star \psi_t$$

for  $\psi_t \in G(\mathcal{A}_+)_t$ , the group of sections that are regular in  $z$  and  $t$ . I write this equivalence as  $\gamma_t'^* \omega \sim \gamma_t^* \omega$ .

*Remark 3.* This gauge equivalence is the same as the statement  $\gamma_t' = \gamma_t \star \psi_t$ , and specifically,  $\gamma_{t-}' = \gamma_{t-}$ . The gauge equivalence on the connection classifies pullbacks by the counterterms of the corresponding sections.

This condition defines a property on the pullbacks of the connection called equisingularity. To see this formally, define:

**Definition 9.** The pullback  $\gamma_t^* \omega$  along on  $P^* \rightarrow B^*$  is equisingular when pulled back to the bundle  $P^* \rightarrow \Delta^*$  if and only if

- $\omega$  is equisingular under the  $\mathbb{C}^\times$  action on the section of  $P^* \rightarrow B^*$ .
- For every pair of sections  $\sigma, \sigma'$  of the  $B \rightarrow \Delta^*$  bundle,  $\sigma(0) = \sigma'(0)$ , the corresponding pull backs of the connection  $\omega, \sigma^*(\gamma_t^* \omega)$  and  $\sigma'^*(\gamma_t^* \omega)$  are equivalent under the action of  $G(\mathcal{A}_+)$ .

*Remark 4.* If  $\gamma(z)$  satisfies (4), Ebrahimi-Fard and Manchon show that

$$\text{Res}(b_{\gamma_t}) = t^Y(\beta(\gamma_t)) = t^Y(\beta(\gamma_{-}^{*-1})) . \quad (9)$$

The following summarizes the important properties of equisingular connections.

**Proposition 4.6.** *The following statements are equivalent:*

1.  $\gamma_t^* \omega$  is an equisingular connection
2. Let  $D\gamma(z, t) = \omega$ . The counterterm is independent of the renormalization mass parameter

$$\frac{d}{dt} \gamma_{-}(z, t) = 0 . \quad (10)$$

*This is the same as equation (4).*

3. Write

$$\gamma_t^* \omega = a(z, 1)dz + b(z, 1)\frac{dt}{t}$$

*We can write*

$$b(z, 1) = \sum_{i=-1}^{\infty} \alpha_i z^i$$

*where  $\alpha_i \in \mathfrak{g}(\mathbb{C})$*

4. The coefficient  $\alpha_{-1} = \beta(\gamma)$  determines the pullback of the connection  $\gamma^* \omega$  up to a  $G(\mathcal{A}_+)$  equivalence.

The following is a generalization of Connes and Marcolli's main theorem in [7]:

**Theorem 4.7.** *Let  $\omega$  be a connection on  $P^* \rightarrow B^*$  defined on the pullbacks as  $\gamma_{t^z}^* \omega = D\gamma_{t^z}$ . For  $\gamma$  satisfying (10), the pullbacks are defined by  $\tilde{R}(\gamma_{t^z-})$  up to the equivalence class defined by  $G(\mathcal{A}_+)$  and is determined by the  $\beta$ -function of the Lagrangian.*

Choosing a regularization scheme for a Lagrangian fixes a section of the renormalization bundle. From Proposition 4.6, two regularization schemes for a Lagrangian lead to the same  $\beta$ -function if and only if they are gauge equivalent under the group  $G(\mathcal{A}_+)$ .

**Example 3.** A Lagrangian regularized under dimensional regularization and  $\zeta$ -function regularization satisfy the differential equation (4) and have the same  $\beta$ -function. Therefore

$$(\gamma_{\dim})_t^* \omega \sim (\gamma_\zeta)_t^* \omega .$$

In fact, the evaluations of a  $\zeta$ -function regularization and dimensional regularization on Feynman diagrams differs only by multiplication by a holomorphic function in  $z$ .

The global connection gives a way of translating between different regularization schemes that are not gauge equivalent. That is, it defines a relationship between two different  $\beta$ -functions. In fact, defining a global connection  $\omega$  on this bundle, one can further view regularized Lagrangians that do not satisfy the differential equation (4), which can be represented by a section  $\gamma_t(z, 1)$  that does not satisfy (10), as gauge fields. Even though the  $\beta$ -function is not defined for these sections, they are equivalent to the sections for which a  $\beta$ -function is defined by the gauge group  $G(\mathcal{A})$ .

A second implication of the global connection is that two different Lagrangians that share the same Hopf algebra are now sections that differ only by a gauge transformation.

**Example 4.** The Lagrangians for a QFT in different gravitational settings have different values for  $\phi$ ,  $m$  and  $g$ , given the different curvatures caused by the background gravitation. These Lagrangians represent QFTs in different coordinate patches of a curved universe. Allowing the  $\phi$ ,  $m$  and  $g$  to vary according to the necessary change of coordinates of a particular curved background manifold, allows one to relate the  $\beta$ -function in all coordinate patches to each other simultaneously. Furthermore, if all the Lagrangians in this family are regularized in the same way, say with dimensional regularization, the corresponding section of the renormalization bundle all satisfy the differential equation (4). Then the gauge transformation holomorphic in  $z$ , and the two  $\beta$ -functions in each coordinate patch can be related to each other. This shows that BPHZ renormalization is consistent over a curved space-time manifold, and suggests that this renormalization bundle can be constructed as a bundle over a curved background space as opposed to the flat example constructed by Connes and Marcolli in [7]. This global  $\beta$ -function can be found via  $\zeta$ -function regularization, but its construction is beyond the scope of this paper.

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